The Structure of Cones of Matrices

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ABSTRACT

If K is a cone in R_n we let $\Gamma(K)$ denote the cone in the space M_n of $n\times n$ matrices consisting of all A such that $AK\subseteq K$. We show first that $\Gamma(K)$ is indecomposable if and only if K is indecomposable. Next we let $\Gamma(K)^*$ be the dual of $\Gamma(K)$. Then we show that $\Gamma(K)=\Gamma(K)^*$ if and only if K is the image of the nonnegative orthant under an orthogonal transformation.

1. PRELIMINARIES

We shall be dealing with cones and partial orders in the space R_n of n dimensional column vectors and in the space M_n of $n\times n$ real matrices. All topological notions refer to the usual topology in these spaces. The term *cone* is used in a restricted sense, as is seen in the next definition (which applies to both R_n and M_n).

Definition 1.1. Let V be a finite dimensional real vector space. A set $K\subseteq V$ is a cone if

(1) $x,y\in K$, $\alpha,\beta\geqslant 0$ implies $\alpha x+\beta y\in K$;
(2) K is closed;
(3) $K\cap(-K)=\{0\}$;
(4) K^0, the interior of K, is nonempty.

Frequently, the term cone is used for a set satisfying (1). If it also satisfies (3) it is called *pointed*, and if it satisfies (4) it is called *full*. In finite dimensional spaces a cone is full if and only if it is *reproducing*, that is, $K-K=V$.

A cone K in V induces a partial order if we define $x \leqslant y$ to mean $y-x \in K$. Then $x<y$ shall mean $y-x \in K$ and $y \neq x$, while $x \ll y$ shall mean $y-x \in K^{\circ}$. Certain distinguished subsets of K, called *faces*, are central to our work.

DEFINITION 1.2. Let K be a cone in V.

(1) A subset $F \subseteq K$ is a *face* of K, denoted $F \trianglelefteq K$, iff
(a) $x, y \in F$, α, $\beta \geqslant 0$ imply $\alpha x+\beta y \in F$,
(b) $0 \leqslant x \leqslant y$ and $y \in F$ imply $x \in F$.
(2) If $F \trianglelefteq K$, then $F-F=V_1$ is a subspace of V called the *span* of F. Its dimension is denoted $\dim F$.
(3) An *extremal* of K is a face of dimension 1. The set of all extremals of K is denoted $\operatorname{Ext} K$.
(4) If $S \subseteq K$, then the face generated by S, which is the least face containing S, is denoted by $\varphi(S)$, or $\varphi(x)$ if $S=\{x\}$.

Let K be a cone in R_n. Then the dual cone is the set

$$K^{*}=\{y \mid y^{T} x \geqslant 0 \quad \text{for all } x \in K\}.$$

It is easily verified that K^{*} is a cone in the sense of (1.1).

As is well known, a partial order in R_n generated by a full cone induces in M_n a partial order. In fact, two such partial orders are given in the next definition.

DEFINITION 1.3. Let K be a cone in R_n.

(1) $\Gamma(K)=\{A \in M_n \mid AK \subseteq K\}$.
(2) $\Delta(K)=\operatorname{cl} \operatorname{conv}\{x y^{T} \mid x \in K, y \in K^{*}\}$.

Note that in (2) cl denotes the closure and conv the convex hull of the corresponding sets. It is known (cf. Berman and Gaiha [1]) that $\Gamma(K)^{*}=\Delta(K^{*})$.

2. INDECOMPOSABLE CONES

Following Loewy and Schneider [3] we say that a cone K is a *direct sum* of subsets K_1 and K_2 (and we write $K=K_1 \oplus K_2$) if

(a) $\operatorname{span} K_1 \cap \operatorname{span} K_2=\{0\}$,
(b) $K=K_1+K_2$.

DEFINITION 2.1. The cone K is called *decomposable* iff there exist nonzero subsets K_1 and K_2 such that $K=K_1\oplus K_2$. Otherwise K is called *indecomposable*.

PROPOSITION 2.1. *If $K=K_1\oplus K_2$ and if $F\trianglelefteq K$, then there are $F_i\trianglelefteq K$, $F_i\subseteq K_i$, such that $F=F_1\oplus F_2$.*

REMARK. *K_i is a face of K, and K_i is a cone in* span K_i. *Thus we may write $F_i\trianglelefteq K_i$ (cf.* [3]).

Proof. Choose x so that $F=\varphi(x)$. Then x has a unique representation $x=x_1+x_2$ with $x_i\in K_i$. Let $F_i=\varphi(x_i)$. Then $F_i\trianglelefteq K_i$ and $\operatorname{span}F_1\cap\operatorname{span}F_2=\{0\}$. Now let $y\in F$ and suppose $0\leqslant \alpha y\leqslant x=x_1+x_2$. We know that $y=y_1+y_2$ with $y_i\in K_i$ and that $z=(x_1-\alpha y_1)+(x_2-\alpha y_2)\in K$. By the uniqueness of representations it follows that $x_i-\alpha y_i\in K_i$, whence $y_i\in F_i$, since $F_i\trianglelefteq K_i$. Thus $F=F_1\oplus F_2$. ∎

THEOREM 2.1. *Let K be a cone in R_n. The following are equivalent:*

(a) *K is indecomposable,*
(b) *K^* is indecomposable,*
(c) Γ(K) is indecomposable,
(d) *$\Delta(K)$ is indecomposable.*

LEMMA 2.1. *K is indecomposable iff K^* is indecomposable.*

Proof. By theorem (3.3) of Loewy and Schneider [3], K is indecomposable iff $I\in\operatorname{Ext}\Gamma(K)$. Also $A\in\Gamma(K)$ iff $A^T\in\Gamma(K^*)$. Hence $I\in\operatorname{Ext}\Gamma(K)$ iff $I\in\operatorname{Ext}\Gamma(K^*)$. ∎

LEMMA 2.2. *K is indecomposable iff $\Gamma(K)$ is indecomposable.*

Proof. Suppose $\Gamma(K)$ is decomposable.. Then by Lemma 2.1 $\Gamma(K)^*=\Delta(K^*)$ is decomposable, say $\Delta(K^*)=\Delta_1\oplus\Delta_2$. Clearly each Δ_i contains at least one rank 1 extremal. For each $y\in\operatorname{Ext}K^*$, $y\neq 0$, put

$$S_i(y)=\{x|x\in\operatorname{Ext}K,\ yx^T\in\Delta_i\},\qquad i=1,2.$$

Suppose there is a y such that both $S_1(y)\neq\{0\}$ and $S_2(y)\neq\{0\}$. Then let $F_i=\varphi(S_i)\trianglelefteq K$. Since each yx^T is in $\operatorname{Ext}\Delta(K^*)$ when $x\in\operatorname{Ext}K$, it follows that $S_1\cup S_2=\operatorname{Ext}K$. Thus $F_1+F_2=K$. Suppose $z\in\operatorname{span}F_1\cap\operatorname{span}F_2$. Let $z=z_1-$

$z_2 = w_1 - w_2$, where $z_1, z_2 \in F_1$, $w_1, w_2 \in F_2$. Then $y(z_1 - z_2)^T = yz_1^T - yz_2^T \in \operatorname{span}\Delta_1$, while $y(w_1 - w_2)^T \in \operatorname{span}\Delta_2$. Thus $yz^T \in \operatorname{span}\Delta_1 \cap \operatorname{span}\Delta_2 = \{0\}$, so $z = 0$ since $y^T \neq 0$. Hence in this case $F = F_1 \oplus F_2$. On the other hand, if there is no such y, then for each $y \in \operatorname{Ext} K^*$ ($y \neq 0$), either $S_1(y) = \{0\}$ or $S_2(y) = \{0\}$. Now let

$$T_1 = \{ y \mid y \in \operatorname{Ext} K^*, y \neq 0, S_2(y) = \{0\} \},$$

$$T_2 = \{ y \mid y \in \operatorname{Ext} K^*, y \neq 0, S_1(y) = \{0\} \}.$$

Then T_1 and T_2 are both nonempty, and we put $G_i = \varphi(T_i)$, $i = 1,2$. Thus $T_1 \cup T_2 = \operatorname{Ext} K^*$, and so $G_1 + G_2 = K^*$. Suppose $w \in \operatorname{span} G_1 \cap \operatorname{span} G_2$. Then $w = w_1 - w_2 = z_1 - z_2$, where $w_1, w_2 \in G_1$, $z_1, z_2 \in G_2$. Let $x \in \operatorname{Ext} K$, $x \neq 0$. Then

$$(z_1 - z_2)x^T = wx^T = (w_1 - w_2)x^T \in \operatorname{span}\Delta_1 \cap \operatorname{span}\Delta_2.$$

Thus as before $w = 0$ and $K^* = G_1 \oplus G_2$. So by Lemma 2.1, K is also decomposable.

Now suppose $K = K_1 \oplus K_2$. Put

$$\Gamma_{ij} = \{ A \mid A \in \Gamma(K), AK_i \subseteq K_j, AK_l \subset \{0\} \text{ for } l \neq i \}.$$

It is easy to see that $\Gamma(K) = \Gamma_{11} + \Gamma_{12} + \Gamma_{21} + \Gamma_{22}$ and that $\Gamma_{ij} \trianglelefteq \Gamma(K)$ for each i and j. Put $\Theta = \varphi(\Gamma_{12} + \Gamma_{21} + \Gamma_{22}) \trianglelefteq \Gamma(K)$. Then $\Gamma_{11} + \Theta = \Gamma(K)$. Let $A \in \Gamma_{11} \cap \Theta$, $x \in K$. Then $x = x_1 + x_2$. Since $A \in \Gamma_{11}$, $Ax_2 = 0$. But also one can find $B_1 \in \Gamma_{12}$, $B_2 \in \Gamma_{21}$, $B_3 \in \Gamma_{22}$ such that $0 \leqslant A \leqslant B_1 + B_2 + B_3$. Then $0 \leqslant Ax_1 \leqslant B_1x_1 + B_2x_1 + B_3x_1 = B_1x_1 \in K_2$. Thus $Ax_1 \in K_1 \cap K_2 = \{0\}$. Hence $Ax = 0$. Since K is full, $A = 0$. Therefore $\Gamma(K)$ is decomposable. ■

Proof of Theorem 2.1. Since $\Delta(K) = \Gamma(K^*)^*$, the theorem follows from the two lemmas. ■

3. SELF-DUAL CONES

DEFINITION 3.1. Let K be a cone in the real vector space V, and let $\dim V = n$.

(1) K is *polyhedral* if it has a finite number of extremals.
(2) K is *simplicial* if it has $n = \dim V$ extremals.
(3) K is *self-dual* if $K^* = K$.

REMARK 3.1. *It is a well known result that* K *is simplicial if and only if it induces a lattice order in* R_n [2, p. 354].

DEFINITION 3.2. Subsequently P will denote the nonnegative orthant, that is, the cone of all vectors x such that each component of x is nonnegative.

The next proposition generalizes a lemma communicated to us by B. Levinger at the Auburn Matrix Theory Conference.

PROPOSITION 3.1. *Let* I *denote the identity matrix.* $I \in \Delta(K)$ *if and only if* K *is simplicial.*

Proof. If K is simplicial, then there is a nonsingular $A \in M_n$ such that $P = AK$. Then $\Gamma(K) = A^{-1}\Gamma(P)A$ and $\Delta(K) = A^{-1}\Delta(P)A$. Since $\Gamma(P) = \Delta(P)$, it follows that $\Delta(K) = \Gamma(K)$.

Conversely, suppose $I \in \Delta(K)$. Then there is a sequence $\{\Sigma_n\} \subseteq \Delta(K) \subseteq \Gamma(K)$ of the form

$$\Sigma_n = \sum_{i=1}^{k_n} x_{in} y_{in}^T, \qquad x_{in} \in K, \quad y_{in} \in K^*$$

such that $\Sigma_n \to I$. For any x, $y \in R_n$ we define $\sup\{x,y\}$ as follows. Let

$$\sigma_n = \sum_{i=1}^{k_n} \max\{(y_{in}^T x), (y_{in}^T y)\} x_{in}.$$

We wish to show that $\{\sigma_n\}$ is a bounded sequence. Let μ_1 be a norm which is additive on K [i.e., $x_1, x_2 \in K$ implies $\mu_1(x_1 + x_2) = \mu_1(x_1) + \mu_1(x_2)$]. Let $e \in K^0$, and let μ be the order unit norm corresponding to e with dual norm μ^D and induced matrix norm λ. Then if $y^T \in K^*$ and $A \in \Gamma(K)$, we have

$$\mu^D(y^T) = y^T e, \qquad \lambda(A) = \mu(Ae).$$

Since μ_1 and μ are equivalent, there is an $\alpha > 0$ such that for all $x \in R_n$, $\mu_1(x) \leqslant \alpha\mu(x)$. Since for $y \neq 0$

$$xy^T = [\mu^D(y^T)x]\left[(\mu^D y^T)^{-1} y^T\right],$$

we may assume that for all i and all n $\mu^D(y_{in}^T)=1$. Finally we let $M=\max\{\mu(x),\mu(y)\}$. Then we have

$$\begin{aligned}
\mu_1(\sigma_n) &\leqslant \sum_{i=1}^{k_n} |\max\{(y_{in}^T x),(y_{in}^T y)\}|\,\mu_1(x_{in}) \\
&\leqslant M\sum_{i=1}^{k_n}\mu_1(x_{in}) = M\mu_1\left(\sum_{i=1}^{k_n} x_{in}\right) \\
&\leqslant M\alpha\mu\left(\sum_{i=1}^{k_n} x_{in}\right) = M\alpha\mu\left(\sum_{i=1}^{k_n}(y_{in}^T e)x_{in}\right) \\
&= M\alpha\mu(\Sigma_n e) = M\alpha\lambda(\Sigma_n),
\end{aligned}$$

which is bounded, since $\Sigma_n \to I$ implies $\lambda(\Sigma_n)\to 1$. Then $\{\sigma_n\}$ is a bounded sequence of vectors, and passing to a subsequence if necessary we obtain a limit $\sigma=\sigma(x,y)$. Note that (taking the limit via the subsequence) we have

$$\begin{aligned}
\sigma - x &= \lim_n \sum_{i=1}^{k_n} \max\{(y_{in}^T x),(y_{in}^T y)\}x_{in} - \lim_n \sum_{i=1}^{k_n}(y_{in}^T x)x_{in} \\
&= \lim_n \sum_{i=1}^{k_n}\left[\max\{(y_{in}^T x),(y_{in}^T y)\} - (y_{in}^T x)\right]x_{in} \geqslant 0.
\end{aligned}$$

A similar argument also shows that $\sigma \geqslant y$.

Now suppose that $z\geqslant x$ and $z \geqslant y$. Then $y_{in}^T(z-x)\geqslant 0$ and $y_{in}^T(z-y)\geqslant 0$ for all i and n, whence

$$y_{in}^T z - \max\{(y_{in}^T x),(y_{in}^T y)\} \geqslant 0.$$

Therefore $z-\sigma\geqslant 0$.

If σ' is the limit of any other subsequence of the original σ_n, then the above argument remains valid for σ' and its corresponding subsequence. But then both $\sigma\geqslant\sigma'$ and $\sigma'\geqslant\sigma$, whence $\sigma=\sigma'$. Thus K is simplicial. ■

THEOREM 3.1. *$\Gamma(K)=\Gamma(K)^*$ if and only if $K=QP$ for some orthogonal matrix Q.*

We shall obtain the proof as a series of lemmas.

LEMMA 3.1. *If $\Gamma(K)=\Gamma(K)^*$, then $K=K^*$.*

Proof. If $A \in \Gamma(K)$, $B \in \Gamma(K)^*$, then we may take the inner product as $(B,A) = \operatorname{tr} B^T A$.

Suppose $y \in K^*$ but $y \notin K$. Then there is a $z \in K^*$ such that $z^T y < 0$ and $\forall x \geqslant 0$, $z^T x \geqslant 0$. Since $\Gamma(K) = \Gamma(K)^*$, we have $A \geqslant 0$ if and only if $A^T \geqslant 0$. Let $u \in K^0$. Then $zu^T \in \Gamma(K)^* = \Delta(K^*)$ and $uy^T \in \Gamma(K)$, whence $yu^T = (uy^T)^T \in \Gamma(K)$ as well. Thus

$$0 \leqslant (zu^T, yu^T) = \operatorname{tr} uz^T yu^T = (z^T y)(u^T u) < 0.$$

Therefore $K^* \subseteq K$.

If now $x \in K$ but $x \notin K^*$, pick $w \in K = K^{**}$ such that $w^T x < 0$ and $\forall y \in K^*$, $y^T w \geqslant 0$. Let $z \in K^*$, $z \neq 0$. Then $xz^T \in \Gamma(K)$, whence $zx^T \in \Gamma(K)$ and $zw^T \in \Gamma(K)^*$. Thus

$$0 \leqslant (zw^T, zx^T) = (z^T z)(w^T x) < 0.$$

Therefore $K \subseteq K^*$ and equality holds. ■

LEMMA 3.2. *If* $\Gamma(K) = \Gamma(K)^*$, *then* K *is simplicial.*

Proof. If $\Gamma(K) = \Gamma(K)^* = \Delta(K^*)$, then $I \in \Delta(K^*)$. Therefore by Proposition (3.1), K^* is simplicial and so is K. ■

PROPOSITION 3.2. *K is a simplicial self-dual cone if and only if $K = QP$ for some orthogonal matrix Q.*

Proof. If $K = QP$, then $K^* = (Q^T)^{-1} P = QP = K$, and so K is self-dual and simplicial. Conversely, suppose $K = K^*$ and $K = AP$, where A is nonsingular. Then $P = P^* = A^T K^* = A^T K$. Thus $A^T A$ is a one to one map of P onto itself. Thus $(A^T A)^{-1} \in \Gamma(P)$ also. Consequently, $A^T A = RD$, where R is a permutation matrix and D is a matrix with positive diagonal elements. In particular $A^T A$ is positive definite symmetric. But since $A^T A = RD$ is positive definite, its principal minors are all positive. Thus RD is a diagonal matrix so $R = I$. Let $B = AD^{-1/2}$. Then B maps P onto K as well, and $B^T B = I$. Thus B is orthogonal. ■

Proof of theorem. If $K = QP$ where Q is orthogonal, then as in the proof of Proposition (3.1) we have that $\Delta(K) = \Gamma(K)$. But also $\Delta(K) = \Delta(K^*)$, so $\Gamma(K) = \Gamma(K)^* = \Delta(K^*)$. Conversely, if $\Gamma(K) = \Gamma(K)^*$, then by Lemmas (3.1) and (3.2), K is a self-dual simplex and we are done by Proposition (3.2). ■

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